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## APPROXIMATION OF LYAPUNOV EXPONENTS OF NON-LINEAR STOCHASTIC DIFFERENTIAL SYSTEMS

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## Abstract

We consider non-linear stochastic differential systems defined either on a compact orientable manifold, or on  $\mathbb{R}^d$ ; under our hypotheses, their Lyapunov exponents are deterministic. We propose an efficient algorithm of numerical computation of these exponents, and we give a theoretical estimate for the approximation error.

The method is based upon the discretization of the linearized stochastic flows of diffeomorphisms generated by the differential systems.

Results of numerical experiments are also presented.

# APPROXIMATION DES EXPOSANTS DE LYAPUNOV DE SYSTEMES DIFFERENTIELS STOCHASTIQUES NON LINEAIRES

## Résumé

Nous considérons des systèmes différentiels stochastiques non linéaires, définis soit sur une variété compacte orientable, soit sur  $\mathbb{R}^d$ ; sous nos hypothèses, leurs exposants de Lyapounov sont déterministes. Nous en proposons un algorithme de calcul numérique efficace, dont nous donnons la vitesse de convergence.

La méthode est fondée sur la discrétisation des flots stochastiques de difféomorphismes linéarisés engendrés par les systèmes différentiels.

Nous présentons aussi des résultats de tests numériques.

# 1 Introduction

The Lyapunov exponents of a stochastic dynamical system enable to study its stability. A survey of this important theory, for linear and non linear systems, may be found in [3], and in Arnold[1] (we will use the notations of this last reference).

From an applied point of view, it is necessary to compute the numerical value of the Lyapunov exponents. For the linear case, an algorithm has been proposed to compute the upper one, and its theoretical convergence rate given, in Talay [12] (where an application to an industrial problem is also described). In practice, most models are non-linear, so it seems interesting to give numerical tools of studying the stability in that context. This is the aim of this paper. The main difference between the two situations is that, for a non linear system, the exponents are expressed in terms of the linearized flow of diffeomorphisms generated by the system ; the algorithm proposed here is based upon the time discretization of this flow. We give a convergence rate of the method in terms of the discretization step. We will also see that one can approximate all the exponents (not only the top one) as well.

We will limit ourselves to the case where the state space is  $\mathbb{R}^d$ , or a smooth compact orientable manifold. The reasons for this limitation are multiple : first, we need the existence of a stochastic flow of diffeomorphisms associated to the stochastic differential equation ; second, the approximation algorithm of the solution of the system has a reasonable complexity in this framework (either one discretizes a system given in euclidian coordinates, or one just needs to define a finite number of charts satisfying a given property stated below) ; third, it is possible to write natural explicit conditions on the coefficients of the system ensuring the ergodicity of the solution ; and, finally, we need to establish a technical result on the solution of a parabolic P.D.E. on the projective bundle of the state space, and our method could fail for non compact manifolds without supposing very stringent conditions on the coefficients of the system.

As we will see, the case of compact manifolds leads to particular developments ; first, we propose an algorithm which takes into account the geometry of the state space, and is nevertheless realistic from a practical point of view ; second, some details of the proof giving the convergence rate of the method are simpler than in the  $\mathbb{R}^d$  case.

We have chosen not to separate the two situations in the presentation. Thus, we avoid a large amount of repetitions, and also it seems easier to distinguish where the compactness plays a role.

In Sections 2 and 3, we present the discretization method of the linearized system, which will permit us to approximate the upper Lyapunov exponent ; in Section 4, we state the hypotheses under which we will establish the convergence rate of our algorithm ; in Section 5, we prove that the Markov chain defined by the discretization admits a Lyapunov exponent ; in Section 6, we prove that our method is a first order approximation ; in Section 7, we give the proof of a technical Lemma used in Section 6 ; in Section 8, we explain how the method can be extended to get a first order approximation of any

exponent of the spectrum ; in Section 9, we present the results of numerical trials, for examples of systems on  $\mathbb{R}^d$  and on a sphere.

We will use the results due to Arnold & San Martin, Baxendale and Caverhill for the continuous time processes (cf. the papers in [3]), and the results due to Bougerol ([5] and [6]) for the discrete time approximating process.

## 2 Discretization of systems on $\mathbb{R}^d$

We consider a Stochastic differential system on  $\mathbb{R}^d$  defined by the equation :

$$\begin{aligned} dx_t &= A(x_t)dt + \sum_{j=1}^r B_j(x_t) \circ dW_t^j \\ x_0 &= x \end{aligned} \quad (1)$$

with smooth vector fields  $A$  and  $B_j$ 's, and its linearized system defined on  $\mathbb{R}^d \times \mathbb{R}^d$  by

$$\begin{aligned} dx_t &= A(x_t)dt + \sum_{j=1}^r B_j(x_t) \circ dW_t^j \\ dv_t &= A'(x_t)v_t dt + \sum_{j=1}^r B'_j(x_t)v_t \circ dW_t^j \\ x_0 &= x, v_0 = v \end{aligned} \quad (2)$$

We will suppose :

**(H1)** *The vectors fields  $A$  and  $B_j$  ( $j = 1 \dots r$ ) are of class  $\mathcal{C}^\infty$  and have bounded derivatives (for all order of derivation) ; the vector fields  $B_j$  ( $j = 1 \dots r$ ) are bounded.*

We are going to define a Markov chain  $(\bar{x}_p^h, \bar{v}_p^h)$ , which can be easily simulated on a computer, and approximates the solution of the system (2).

### 2.1 Discretization scheme

We begin by the following remark : under (H1), for any random variable  $U$  with a compact supported law  $P_U$ , there exists  $h_0(P_U) > 0$  satisfying : for any  $h \leq h_0(P_U)$ , if

$$\tilde{A}(x) = A(x) + \frac{1}{2} \sum_{j=1}^r B'_j(x) B_j(x)$$

then :

$$\frac{1}{2} \leq \|Id + \tilde{A}'(x)h + \sum_{j=1}^r B'_j(x)U\sqrt{h}\| \leq 2, \quad a.s$$

Then we take an  $h$  in  $\mathbb{R}_+$  and a family  $(U_{p+1}^j)$  of random variables which will be supposed to satisfy the following requirement :

- (HU) (i) the  $(U_{p+1}^j)$ 's are i.i.d., and the following conditions on the moments are fulfilled :  $E[U_{p+1}^j] = E[U_{p+1}^j]^3 = 0$ ,  $E[U_{p+1}^j]^2 = 1$  ;
- (ii) the step-size  $h$  is less than  $h_0(P_U)$  ;
- (iii) the common law  $P_U$  of the  $(U_{p+1}^j)$ 's has a continuous density w.r.t. the Lebesgue measure, whose support contains an open interval including 0 and is compact.

Let take a deterministic initial value  $(x_0, v_0)$  in  $\mathbb{R}^d \times \mathbb{R}^d$  and set :

$$\begin{aligned}\bar{x}_{p+1}^h &= \bar{x}_p^h + \bar{A}(\bar{x}_p^h)h + \sum_{j=1}^r B_j(\bar{x}_p^h)U_{p+1}^j\sqrt{h} \\ \bar{M}_{p+1}^h &= Id + \bar{A}'(\bar{x}_p^h)h + \sum_{j=1}^r B_j'(\bar{x}_p^h)U_{p+1}^j\sqrt{h} \\ \bar{v}_{p+1}^h &= \bar{M}_{p+1}^h \bar{v}_p^h\end{aligned}\tag{3}$$

This system describes the passage from  $(\bar{x}_p^h, \bar{v}_p^h)$  to  $(\bar{x}_{p+1}^h, \bar{v}_{p+1}^h)$  using an Euler approximation of step-size  $h$  of the process  $(x_t, v_t)$  in  $\mathbb{R}^d \times \mathbb{R}^d$ .

## 2.2 Remark

The condition (HU) does not allow the simulation of a gaussian law, whereas one could expect that  $U_{p+1}^j\sqrt{h}$  would be an approximation of  $W_{(p+1)h}^j - W_{ph}^j$ .

This is not a limitation, neither numerically (the computers prefer to simulate compact-supported laws !), nor theoretically : the rate of convergence of the algorithm would not be better if the exact gaussian law would be simulated (cf. also the approximation of the law of a diffusion process (Talay [11])).

## 3 Discretization of systems on a compact manifold

For a system on a compact manifold, we can write a version of the previous algorithm. This version is very simple to implement (at least if one can define explicit coordinate maps).

Let us consider a  $d$ -dimensional  $\mathcal{C}^\infty$  compact manifold  $\mathcal{M}$ , and  $A, B_j$  ( $j = 1, \dots, r$ ) vector fields on  $\mathcal{M}$ . We also consider a  $r$ -dimensional standard Wiener process  $(W_t)$ .

In this framework, we reformulate (H1) as follows :

(H1) The vectors fields  $A$  and  $B_j$  ( $j = 1 \dots r$ ) are of class  $\mathcal{C}^\infty$ .

We will deal with the stochastic differential system in the Stratonovich sense on  $\mathcal{M}$  :

$$dx_t = A(x_t)dt + \sum_{j=1}^r B_j(x_t) \circ dW_t^j \quad (4)$$

This system defines a stochastic flow of diffeomorphisms  $(x_t(x))$  (cf. Ikeda & Watanabe [9] e.g.) ; if  $Tx_t(x) : T_x\mathcal{M} \rightarrow T_{x_t(x)}\mathcal{M}$  is the linear part of  $x_t$  at  $x$ , and if the vector fields  $TA, TB_j$  are the linearizations of  $A, B_j$ , then the mapping  $Tx_t$  from  $T\mathcal{M}$  to  $T\mathcal{M}$  defined by  $(x, v) \rightarrow (x_t(x), Tx_t(x)v)$  is a flow on the tangent bundle  $T\mathcal{M}$ , generated by the system

$$dT_x t = TA(Tx_t)dt + \sum_{j=1}^r TB_j(Tx_t) \circ dW_t^j \quad (5)$$

Our objective is to discretize this system.

### 3.1 Preliminary

Let us begin by an elementary remark.

**Proposition 3.1** *We may choose an atlas  $\mathcal{A}$  on  $\mathcal{M}$  such that :*

- (i)  *$\mathcal{A}$  has a finite number of charts  $(\phi, \text{Dom}(\phi))$  ;*
- (ii) *let  $(\phi, \text{Dom}(\phi))$  an arbitrary chart, and  $\alpha$  (resp.  $\beta_j$ ) the expression in local coordinates of the vector field  $A$  (resp.  $B_j$ ) ; then, for any non void multi-index  $I$ , and any  $1 \leq i \leq d, 1 \leq j \leq r$ , the derivative  $\partial_I \alpha^i$  (resp.  $\partial_I \beta_j^i$ ) is bounded on  $\text{Val}(\phi)$  by a constant depending only on  $I$  (not on  $\phi$ ) ;*
- (iii) *for each  $\phi$ ,  $\text{Val}(\phi)$  is included in the ball  $B(\phi(x), 1)$  and is convex ;*
- (iv)  *$\exists R > 0$  such that :  $\forall x \in \mathcal{M}, \exists (\phi, \text{Dom}(\phi)) \in \mathcal{A} : B(\phi(x), R) \subset \text{Val}(\phi)$*

*An atlas satisfying the properties (i), (ii), (iii) and (iv) will be called a large charts atlas on  $\mathcal{M}$ .*

**Proof** Let us suppose : for any atlas satisfying (i), (ii), and (iii) (easy to obtain), the following is true :  $\forall (R_n, n \in \mathbb{N})$  such that  $R_n \downarrow 0$ , there exists  $x_n \in \mathcal{M}$ , such that, if  $(\phi, \text{Dom}(\phi))$  is any chart around  $x_n$ ,  $B(\phi(x_n), R_n)$  is not included in  $\text{Val}(\phi)$ . This is impossible : as  $\mathcal{M}$  is compact, a subsequence  $(x_{n_k})$  converges to a limit  $x$  ;  $(\phi, \text{Dom}(\phi))$  being a chart around  $x$ , the continuity of  $\phi$  implies : for  $k$  large enough,  $B(\phi(x_{n_k}), R_{n_k}) \subset \text{Val}(\phi)$ .

□

For any vector-valued function  $\gamma$ , let  $\partial\gamma$  denote the matrix  $[\partial_k \gamma^i]_{i,k}$ .

The previous Proposition implies :

**Corollary 3.2** *For any random variable  $U$  with a compact supported law  $P_U$ , there exists  $h_0(P_U) > 0$  satisfying, for any  $h \leq h_0(P_U) : \forall x \in \mathcal{M}$ , there exists a chart  $(\phi, \text{Dom}(\phi))$  in  $\mathcal{A}$  such that, if  $\alpha, \beta_j$  denote the expression of the vector fields  $A, B_j$  in this chart, and if*

$$\tilde{\alpha} := \alpha + \frac{1}{2} \sum_{j=1}^r (\partial \beta_j) \beta_j$$

then :

$$\phi(x) + \tilde{\alpha}(\phi(x))h + \sum_{j=1}^r \beta_j(\phi(x))U\sqrt{h} \in \text{Val}(\phi) \quad , \quad a.s.$$

and

$$\frac{1}{2} \leq \|Id + \partial \tilde{\alpha}(\phi(x))h + \sum_{j=1}^r \partial \beta_j(\phi(x))U\sqrt{h}\| \leq 2 \quad , \quad a.s.$$

### 3.2 Discretization scheme

The initial value is any deterministic pair  $(x_0, v_0)$  in  $T\mathcal{M}$  ; let us describe the passage from  $(\bar{x}_p^h, \bar{v}_p^h)$  to  $(\bar{x}_{p+1}^h, \bar{v}_{p+1}^h)$  : let  $(\phi_p, \text{Dom}(\phi_p))$  be a large chart around  $\bar{x}_p^h$ , and let  $\alpha_p$  (resp.  $\beta_j^p$ ) be the representation of the vector field  $A$  (resp.  $B_j$ ) in local coordinates ; we set :

$$\tilde{\alpha}_p := \alpha_p + \frac{1}{2} \sum_{j=1}^r (\partial \beta_{j,p}) \beta_{j,p}$$

$\phi_p$  induces a basis  $\epsilon_p$  in  $T_{\bar{x}_p^h} \mathcal{M}$  given by :  $\epsilon_p^j f = \partial_j (f \circ \phi_p^{-1})(\phi_p(\bar{x}_p^h))$  for any smooth function  $f$  on  $\mathcal{M}$  and  $1 \leq j \leq d$ .

Let  $\bar{V}_p^h$  denote the vector of  $\mathbb{R}^d$  whose coordinates in the canonical basis are those of  $\bar{v}_p^h$  in  $\epsilon_p$ , and  $\bar{x}_p^h = \phi_p(\bar{x}_p^h)$ .

Then  $\bar{x}_{p+1}^h$  is computed according to the formula :

$$\bar{x}_{p+1}^h = \phi_p^{-1}(\bar{x}_p^h + \tilde{\alpha}_p(\bar{x}_p^h)h + \sum_{j=1}^r \beta_{j,p}(\bar{x}_p^h)U_{p+1}^j \sqrt{h})$$

where the random variables  $U_{p+1}^j$  will be supposed to satisfy the requirement  $(HU)$  of the previous Section,  $h_0(P_U)$  being defined in the sense of Corollary (3.2).

Next,  $\bar{v}_{p+1}^h$  will be the vector of  $T_{\bar{x}_{p+1}^h} \mathcal{M}$  whose coordinates in the basis  $\epsilon_{p+1}$  are the coordinates in the canonical basis of  $\mathbb{R}^d$  of

$$\bar{V}_{p+1}^h = \bar{M}_{p+1}^h \bar{V}_p^h$$

where

$$\bar{M}_{p+1}^h = Id + \partial \tilde{\alpha}_p(\bar{x}_p^h)h + \sum_{j=1}^r \partial \beta_{j,p}(\bar{x}_p^h)U_{p+1}^j \sqrt{h}$$



### 3.3 Remark

In that case, the restriction to compact-supported laws in the hypothesis (HU) also ensures that for  $h$  small enough

$$\bar{x}_p^h + \alpha_p(\bar{x}_p^h)h + \sum_{j=1}^r \beta_{j,p}(\bar{x}_p^h)U_{p+1}^j \sqrt{h}$$

belongs to  $Val(\phi_p)$ , so that  $\bar{x}_{p+1}^h$  may be defined without any change of chart (in particular, for technical reasons appearing in the proof of the convergence, it is very important that it is obtained with the chart  $(\phi_p, Dom(\phi_p))$ , which is measurable w.r.t. the  $\sigma$ -field generated by  $(\bar{r}_k^j, 0 \leq k \leq p, 1 \leq j \leq r)$ ).

## 4 Basic Hypotheses

In order to ensure the existence of the Lyapunov exponents of the continuous time process and its discretized process, we need to formulate rather technical (but reasonable for the applications) hypotheses ; we had to strengthen these hypotheses to get the rate of convergence of the approximate Lyapunov exponent in terms of the discretization step.

### 4.1 Systems on $\mathbb{R}^d$

Let  $S^{d-1} = \{x \in \mathbb{R}^d ; |x| = 1\}$  be the unit sphere of  $\mathbb{R}^d$ , and  $\mathbb{P}^{d-1}$  be the projective space of  $\mathbb{R}^d$ , i.e. the quotient of  $S^{d-1}$  with respect to the relation :  $u \sim v$  iff  $u = -v$ . In the sequel, for any vector  $v$  in  $\mathbb{R}^d$ ,  $[v]$  will denote the equivalence class of  $v$  in  $\mathbb{P}^{d-1}$ , and  $\mathbb{P}\mathbb{R}^d$  will denote the space  $\mathbb{R}^d \times \mathbb{P}^{d-1}$ .

We know (cf Arnold and San Martin [2]) that the process  $(Px_t(x, [v]))$  on  $\mathbb{P}\mathbb{R}^d$  defined by  $Px_t(x, [v]) = (x_t, [v_t])(x, [v])$  (i.e the process  $(x_t, [v_t])$  with initial condition  $(x, [v])$ ) is solution of the system :

$$\begin{aligned} dPx_t &= PA(Px_t)dt + \sum_{j=1}^r PB_j(Px_t) \circ dW_t^j \\ Px_0 &= (x, [v]) \end{aligned} \tag{6}$$

with  $PC((x, [v])) = (Cx, C'(x)[v] - \langle C'(x)[v], [v] \rangle [v])$  for  $C = A$  or  $B_j$ .

We shall assume that the following hypothesis holds :

**(H2)** (i) The differential operator  $PA + \frac{1}{2} \sum_{j=1}^r (PB_j)^2$  is strongly elliptic.

(ii)  $\exists \beta > 0$ ,  $\exists K$  compact  $\in \mathbb{R}^d$  such that :  $\forall x \in \mathbb{R}^d \setminus K$ ,  $\langle x, A(x) \rangle \leq -\beta|x|^2$

## 4.2 Systems on compact manifolds

For  $x \in \mathcal{M}$ , let  $P_x\mathcal{M}$  be the projective fibre over  $x$ , and  $P\mathcal{M} = \cup_{x \in \mathcal{M}} \{x\} \times P_x\mathcal{M}$  be the projective bundle over  $\mathcal{M}$ . For  $v \in T_x\mathcal{M}$ ,  $[v]$  will denote the equivalence class of  $v$  in  $P_x\mathcal{M}$ .

Let  $Px_t(x, [v])$  be the equivalence class of  $(x_t(x), \frac{T_{x_t(x)}v}{|T_{x_t(x)}v|})$  in  $P\mathcal{M}$ ; this process on  $P\mathcal{M}$  solves a stochastic differential system (cf Arnold & San Martin [2] or Carverhill [7]):

$$\begin{aligned} dPx_t &= PA(Px_t)dt + \sum_{j=1}^r PB_j(Px_t) \circ dW_t^j \\ Px_0 &= (x, [v]) \end{aligned} \quad (7)$$

(We will not need here the explicit expression of the vector fields  $PA, PB_j$ ).

In this framework, the hypothesis (H2) will be :

**(H2)** *the differential operator  $PA + \frac{1}{2} \sum_{j=1}^r (PB_j)^2$  is strongly elliptic.*

## 5 Existence of the upper Lyapunov exponent

In all the sequel, if there is no precision,  $\mathcal{M}$  will denote either  $\mathbb{R}^d$  or a  $C^\infty$  compact manifold. In order to avoid repetitions, we will adopt the notations corresponding to the second case. If the reader is interested in the  $\mathbb{R}^d$  case only, he should substitute  $\mathbb{R}^d$  to each tangent space, and  $v_t$  to  $Tx_t(x)v$ .

Once for all, we give ourselves an atlas  $\mathcal{A}_{\mathbb{P}\mathcal{M}}$  on the projective bundle. When  $\mathcal{M}$  is compact, we just choose any large charts atlas (in the sense of Proposition (3.1)) on this compact manifold. When  $\mathcal{M} = \mathbb{R}^d$ , we choose a large charts atlas on  $\mathbb{P}^{d-1}$  considered as a compact manifold in  $\mathbb{R}^D$  (for some  $D$  whose existence is implied by the Whitney Theorem); we will assume that each coordinate map  $\phi$  satisfies : any partial derivative (of any order) of  $\phi^{-1}$  can be bounded on  $Val(\phi)$ ; the cartesian product of  $\mathbb{R}^d$  with this atlas defines  $\mathcal{A}_{\mathbb{P}\mathcal{M}}$ . We will call it an “extended large charts atlas” on  $\mathbb{P}\mathcal{M}$ .

### 5.1 Ergodicity of the processes $(Px_t)$ and $(P\bar{x}_p^h)$

First consequences of the above hypotheses are the two following Propositions.

Under (H1) and (H2) (cf. Ikeda & Watanabe [9], chapter 5, e.g.), the differential operator

$$PA + \frac{1}{2} \sum_{j=1}^r (PB_j)^2$$

induces on  $P\mathcal{M}$  a Riemannian measure  $d\tau$ .

**Proposition 5.1** *Under (H1) and (H2), the process  $(Px_t)$  on the tangent bundle  $\mathbb{P}\mathcal{M}$  has a unique invariant probability law (henceafter denoted by  $\nu$ ), which has a strictly positive smooth density w.r.t.  $d\tau$ .*

**Proof** First,  $(Px_t)$  is a strong Feller process on a metric space ; second, there exists a strictly positive constant  $C$  such that :  $E|Px_t|^2 < C$  (when  $\mathcal{M} = \mathbb{R}^d$ , this can be checked by using the Ito formula and (H2)(ii)) ; therefore, there exists at least one invariant probability measure (cf. Ethier & Kurtz [8] e.g.).

The conclusion of the Proposition comes from the fact that, under (H1) and (H2), for any deterministic initial condition, the law of  $(Px_t)$  ( $t > 0$ ) has a smooth and strictly positive density with respect to  $d\tau$  (again, cf. Ikeda & Watanabe [9] e.g.).

This ensures that the process  $(Px_t)$  is an ergodic process.

□

In particular, of course this implies that the process  $(x_t)$  itself is ergodic.

Let  $P\bar{x}_p^h$  be the equivalence class of  $(\bar{x}_p^h, \frac{\bar{v}_p^h}{|\bar{v}_p^h|})$  in  $P\mathcal{M}$ .

**Proposition 5.2** *Under (H1), (H2) and (HU), for all  $h$  small enough, the process  $(P\bar{x}_p^h)$  is an ergodic process on  $P\mathcal{M}$ .*

**Proof** When  $\mathcal{M}$  is compact, the same arguments as above show that there exists at least one invariant probability law ; the unicity comes from the fact that, from any starting point on  $\mathcal{M}$ , the process can reach any compact set of  $\mathcal{M}$  in finite time with a strictly positive probability (this results from (H2) and the construction of the scheme).

In the other case, let us consider the measure  $1_K(z)dz \otimes d\rho$ , where  $dz$  is the Lebesgue measure on  $\mathbb{R}^d$ ,  $d\rho$  is the trace of  $d\tau$  on  $\mathbb{P}^{d-1}$ , and  $K$  is the compact set of (H2)(ii).

We deduce from (H1) and (H2)(ii) that, for any starting point  $x$ , the process  $(\bar{x}_p^h)$  reaches the compact set  $K$  in finite time with a strictly positive probability. Let  $\tilde{x}$  be the reached point in  $K$ , and  $K_0$  an arbitrary compact subset of  $K$ .

As the chosen law of the  $U_{p+1}^j$ 's is equivalent to the Lebesgue measure, under (H2), one can show that there exists  $h_0$  independent of  $\tilde{x}$  and  $K_0$  such that, for any  $h < h_0$ ,  $(\bar{x}_p^h)$  reaches  $K_0$  from  $\tilde{x}$  in a finite number of steps with a strictly positive probability. Thus  $(P\bar{x}_p^h)$  reaches any open set of strictly positive  $1_K(z)dz \otimes d\rho$  measure in finite time with a strictly positive probability.

Moreover, (H2)(ii) implies that, for all small enough  $h$ , there exists  $\varepsilon$  positive, such that for all  $x$  outside  $K$  :

$$E|\bar{x}_1^h(x)|^2 \leq |x|^2 - \varepsilon$$

Therefore a result of Tweedie [15] implies the ergodicity of the process  $(P\bar{x}_p^h)$ .

□

## 5.2 Existence of the Lyapunov exponent : continuous time processes

Under the conclusion of the previous Proposition, (H2) is an even too strong condition to ensure that there exists a real number  $\lambda$  such that, for any  $(x, v)$  in  $T\mathcal{M}$  :

$$\lambda = \lim_{t \rightarrow +\infty} \frac{1}{t} \log |Tx_t(x)v| \quad , a.s.$$

(Arnold & San Martin [2]).

The number  $\lambda$  is called the top Lyapunov exponent of the system (2).

Remember that  $\nu$  denotes the unique invariant probability law of  $(x_t, [v_t])$  ; Baxendale [4] gives the following expression for  $\lambda$  :

$$\lambda = \int_{P\mathcal{M}} \psi(\theta) \nu(d\theta)$$

where,  $\nabla$  (resp.  $\mathcal{R}$ ) denoting the Riemannian covariant derivative (resp. the Riemannian curvature tensor) on  $\mathcal{M}$ , for  $x$  in  $\mathcal{M}$  and  $v$  in  $P_x\mathcal{M}$ ,

$$\begin{aligned} \psi(x, [v]) &= \langle v, \nabla(A(x) + \frac{1}{2} \sum_{j=1}^r \nabla B_j(x) B_j(x))v \rangle \\ &+ \frac{1}{2} \sum_{j=1}^r \left[ |\nabla B_j(x)v|^2 - 2 \langle v, \nabla B_j(x)v \rangle^2 + \langle \mathcal{R}(B_j(x), v) B_j(x), v \rangle \right] \end{aligned} \quad (8)$$

In the case  $\mathcal{M} = \mathbb{R}^d$ , the covariant derivative is just the ordinary derivative, and  $\mathcal{R} = 0$ . Thus, under (H1),  $\psi$  is a  $\mathcal{C}^\infty$  bounded function.

Besides, for each multi-index  $I$ , there exists a positive constant  $C_I$  such that, for any chart  $(\phi, \text{Dom}(\phi))$  of the atlas  $\mathcal{A}_{\mathbb{P}\mathcal{M}}$ , the partial derivative  $\partial_I \psi$  of the function  $\psi$  expressed in local coordinates satisfies :

$$|\partial_I \psi \circ \phi^{-1}(y)| < C_I \quad , \quad \forall y \in \text{Val}(\phi)$$

## 5.3 Existence of the Lyapunov exponent : discrete time processes

In either the case  $\mathbb{R}^d$  or the case of a compact manifold, the process  $(\bar{x}_p^h, \bar{M}_p^h)$  in  $\mathcal{M} \times Gl(\mathbb{R}^d)$  is a multiplicative Markov process in the sense of Bougerol [5] : indeed, if  $\bar{R}_p^h$  is its transition operator, for any Borel sets  $E_1$  in  $\mathcal{M}$  and  $E_2$  in  $Gl(\mathbb{R}^d)$  :

$$\bar{R}_p^h((x_0, M) ; E_1 \times E_2 M) = \bar{R}_p^h((x_0, Id) ; E_1 \times E_2) \quad , \quad \forall p \in \mathbb{N} \quad , \quad x_0 \in \mathcal{M} \quad , \quad M \in Gl(\mathbb{R}^d)$$

where :  $E_2 M = \{NM \in Gl(\mathbb{R}^d) ; N \in E_2\}$ .

Besides :

**Proposition 5.3** *Under the hypotheses (H1), (H2), (HU), for any  $h$  small enough :*

(i) *the process  $(\bar{x}_p^h)$  is ergodic ;*

(ii) *let  $\bar{\mu}^h$  be its unique invariant probability law ; then*

$$\sup_{p \leq \frac{1}{h}} E_{\bar{\mu}^h}(\text{Log} \|\bar{M}_p^h\| + \text{Log} \|(\bar{M}_p^h)^{-1}\|) < \infty$$

(iii) *the system  $(\bar{x}_p^h, \bar{M}_p^h)$  is strongly irreducible, in the sense that there does not exist a finite union  $F$  of proper sub-spaces of  $\mathbb{R}^d$  such that, for any integer  $p$ ,  $\bar{M}_p^h F = F$ .*

**Proof** If  $\mathcal{M} = \mathbb{R}^d$ , the result (i) is proved in Talay [13].

Otherwise, we remark that our construction ensures that  $(\bar{x}_p^h)$  is a Feller process on a compact metric space, so that it has at least one invariant measure ; the hypotheses (HU), (H1) and (H2) imply that, from any starting point  $\bar{x}_0^h = x$ , the probability to reach any set of strictly positive  $d\tau$ -measure in a finite time is strictly positive (at each step, the process can go in any direction), therefore  $(\bar{x}_p^h)$  is an ergodic process.

The statement (ii) is satisfied whenever  $h$  is smaller than  $h_0(P_U)$ .

Next, we will use the following consequence of a result due to Bougerol [5] (Proposition 5.2) : if a multiplicative Markov process satisfying (i) and (ii) were not strongly irreducible, then there would exist a continuous finite family  $\{E_1(x), \dots, E_n(x)\}$  of proper sub-spaces of  $\mathbb{R}^d$ , with same dimension, such that, for all  $x$  in  $\mathcal{M}$  :

$$\{\bar{M}_1^h E_1(x), \dots, \bar{M}_1^h E_n(x)\} = \{E_1(x), \dots, E_n(x)\} \quad , a.s.$$

Let us suppose that this property holds. Then we would have :

$$(Id + \partial \hat{\alpha}_0(x_0)h + \sum_{j=1}^r \partial \beta_{j,0}(x_0) U_1^j \sqrt{h})(\cup_{i=1}^n E_i) = \cup_{i=1}^n E_i \quad (9)$$

Taking the expectation of each side of the previous equality, one would get :

$$\partial \hat{\alpha}_0(\cup_{i=1}^n E_i) = \cup_{i=1}^n E_i$$

Multiplying by  $U_{p+1}^j$  before taking the expectation, one would also get :

$$\partial \beta_{j,0}(\cup_{i=1}^n E_i) = \cup_{i=1}^n E_i$$

Thus, using (9),  $F = \cup_{i=1}^n E_i$  would be a proper sub-space of  $\mathbb{R}^d$  such that

$$\partial \alpha_0(F) = \partial \beta_{j,0}(F) = F \quad , \quad \forall j = 1, \dots, d$$

But this would contradict the hypothesis (H1)(iii).

□

A Corollary of Proposition (5.3) (for the proof, cf. Bougerol [6], Section 2) is :

**Theorem 5.4** *Under the hypotheses (H1), (H2), (HU), for any  $h$  small enough, there exists  $\bar{\lambda}^h \in \mathbb{R}$  such that, for any  $x_0$  in  $\mathcal{M}$ , for any  $v_0$  in  $T_{x_0}\mathcal{M}$  :*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log |\bar{v}_N^h(v_0)| = \bar{\lambda}^h, \quad \text{a.s.}$$

The number  $\bar{\lambda}^h$  is called the top Lyapunov exponent of the Markov process  $(\bar{x}_p^h, \bar{v}_p^h)$ .

Numerically, we will compute, for  $N$  large enough, the quantity :

$$\frac{1}{N} \log |\bar{v}_N^h(v_0)| \tag{10}$$

## 6 Approximation error

The theoretical approximation error is given by the

**Theorem 6.1** *Under the hypotheses (H1), (H2), (HU) :  $|\lambda - \bar{\lambda}^h| = \mathcal{O}(h)$ .*

The proof of Theorem (6.1) will be a succession of Lemmas.

**Lemma 6.2**

$$E \log |\bar{M}_{p+1}^h v_0| = E \log |\bar{M}_p^h v_0| + E \psi(P\bar{x}_p^h)h + \bar{r}_p^h h^2 \tag{11}$$

with :  $\psi$  defined in (8),  $|\bar{r}_p^h| \leq C$  (for some constant  $C$  independent from  $p, h$ ).

**Proof** For any Stochastic Differential Equation in  $\mathbb{R}^d$  whose coefficients are smooth with bounded derivatives of any order :

$$dy_t = a(y_t)dt + \sum_{j=1}^r b_j(y_t) \circ dW_t^j$$

and for any smooth bounded function  $f$  with bounded derivatives, one can check that the Euler scheme

$$\bar{y}_{p+1}^h = \bar{y}_p^h + (a(\bar{y}_p^h) + \frac{1}{2} \sum_{j=1}^r \partial b_j(\bar{y}_p^h) b_j(\bar{y}_p^h))h + \sum_{j=1}^r b_j(\bar{y}_p^h) U_{p+1}^j \sqrt{h}$$

satisfies :

$$Ef(\bar{y}_{p+1}^h) = Ef(\bar{y}_p^h) + E\mathcal{L}f(\bar{y}_p^h)h + \bar{z}_{p+1}^h h^2$$

where  $\mathcal{L}$  is the infinitesimal generator of  $(y_t)$ , and  $|\bar{z}_{p+1}^h| \leq C$  ( $C$  independent of  $p, h$ ).

Besides, we know (Baxendale [4]) that, for any initial condition  $(x_0, v_0)$  of the system (5), if  $(Px_t)$  is the solution of (7), then :

$$\frac{d}{dt} E \log |Tx_t(x_0)v_0| = E\psi(Px_t(x_0), [v_0])$$

In other words, if we define  $\phi(x, v) = \log |v|$ , then  $\psi \circ \pi = \mathcal{L}\phi$ , where  $\mathcal{L}$  is the infinitesimal generator of the process  $(T x_t)$  and  $\pi$  is the application from  $T\mathcal{M}$  to  $P\mathcal{M}$  defined by :  $\pi(x, v) = (x, [v])$ .

In the  $\mathbb{R}^d$  case, we just choose  $y_t = (x_t, v_t)$  and  $f(x, v) = \log |v|$ , and we get (11). In the compact case, we express  $(x_t(x), T x_t(x)v)$  and  $\log |v|$  in the local coordinates given by a finite atlas of large charts on  $P\mathcal{M}$ .

□

Iterating (11), one gets :  $E \log |\bar{M}_N^h v_0| = \log |v_0| + \sum_{p=1}^{N-1} E\psi(P\bar{x}_p^h)h + \sum_{p=1}^{N-1} \bar{r}_p^h h^2$ .

Let us divide each term of the previous equality by  $N$  and make  $N$  tend to infinity.

Let  $\bar{\nu}^h$  denote the unique invariant probability law of the process  $(P\bar{x}_p^h)$  ; then the limit

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{p=1}^N E\psi(P\bar{x}_p^h)$$

exists, and is equal to  $\int_{P\mathcal{M}} \psi(\theta) \bar{\nu}^h(d\theta)$ .

Therefore, the conclusion of our Theorem will be implied by the following Lemma, whose hypothesis on  $f$  is weaker than necessary here (bounded would be sufficient, but the boundedness brings no simplification in the proof) :

**Lemma 6.3** *Let  $f$  be a smooth function on  $P\mathcal{M}$  ; when  $\mathcal{M} = \mathbb{R}^d$ , we suppose :*

$$\exists C > 0 \quad , \quad \exists n \in \mathbb{N} \quad , \quad \forall \theta(x, [v]) \in \mathbb{P}\mathbb{R}^d \quad : \quad |f(\theta)| \leq C(1 + |x|^n)$$

Then :

$$|\int_{P\mathcal{M}} f(\theta) \nu(d\theta) - \int_{P\mathcal{M}} f(\theta) \bar{\nu}^h(d\theta)| = \mathcal{O}(h)$$

**Proof** Let  $L$  be the infinitesimal generator of the process  $(P x_t)$  solution of (7) on  $P\mathcal{M}$ , and, for a given smooth function  $f$  on  $P\mathcal{M}$ , and  $\theta \in P\mathcal{M}$ , let  $u(t, \theta) := E_\theta f(P x_t)$ . It is well known that

$$\begin{aligned} \frac{d}{dt} u(t, \theta) &= L u(t, \theta) \\ u(0, \theta) &= f(\theta) \end{aligned} \tag{12}$$

Let us suppose that we have proved (this will be done in the next Section) :

**Lemma 6.4** *Under (H1), (H2) :*

(i) *There exist strictly positive constants  $\Gamma$  and  $\gamma$  such that :*

$$\forall \theta \in P\mathcal{M} \quad : \quad |u(t, \theta) - \int_{P\mathcal{M}} f(s) d\nu(s)| \leq \Gamma \exp(-\gamma t) \tag{13}$$

(ii) for any multi-index  $I$ , there exist strictly positive constants  $\Gamma_I$  and  $\gamma_I$  such that, for any  $\theta = (x, [v])$  in  $P\mathcal{M}$ , any  $z$  in  $Val(\phi_\theta)$ , where  $(\phi_\theta, \text{Dom}(\phi_\theta))$  is a large chart around  $\theta$  in the sense defined at the beginning of Section (5)), the spatial derivative  $\partial_I u(t, \phi_\theta^{-1}(z))$  satisfies :

$$|\partial_I u(t, \phi_\theta^{-1}(z))| \leq \Gamma_I \exp(-\gamma_I t) \quad (14)$$

Then tedious computations (where the remark in Section (3.3) concerning the measurability of the charts around  $\bar{x}_p^h$ , and the property (3.1)(i) of the chosen atlas play a role) show that our construction implies :

$$\forall k, \forall p, \quad Eu(kh, P\bar{x}_{p+1}^h) = Eu(kh, P\bar{x}_p^h) + E Lu(kh, P\bar{x}_p^h)h + \bar{r}_{k,p+1}^h h^2$$

where  $|\bar{r}_{k,p+1}^h| \leq C_1 + C_2 p h$  (for some strictly positive constants  $C_1, C_2$  independent from  $p, h$ ). Then we proceed as in the proof of Lemma 4.4 of Talay [12].

□

## 7 Proof of Lemma (6.4)

In Talay [12] (Lemma 4.3), a similar result has been obtained for processes on the sphere  $S^{d-1}$ ; the construction of the approximating process was different, but the proof can be adapted without difficulty for the case where  $\mathcal{M}$  is compact, just by choosing a finite atlas of large charts on the compact manifold  $P\mathcal{M}$ .

For the case  $\mathbb{R}^d$ , the adaptation is more intricate.

### 7.1 Preliminary inequalities

Let us state a result concerning the partial derivatives of  $u$ . We recall that we work with an extended large charts atlas on  $\mathbb{P}\mathbb{R}^d$  (in the sense of the Section (5)).

**Lemma 7.1** *For any  $t$  and any multi-index  $I$ , there exist strictly positive constants  $C_I(t)$  and  $n_I$  ( $n_I$  independent of  $t$ ) such that, for any  $\theta = (x, [v])$  in  $\mathbb{P}\mathbb{R}^d$ , any  $(x, y)$  in  $Val(\phi_\theta)$ , the spatial derivative  $\partial_I u(t, \phi_\theta^{-1}(x, y))$  satisfies :*

$$|\partial_I u(t, \phi_\theta^{-1}(x, y))| \leq C_I(t)(1 + |x|^{n_I}) \quad (15)$$

**Proof** Having imbedded  $\mathbb{P}^{d-1}$  in  $\mathbb{R}^D$  (for some  $D$ ), we can extend the vector fields  $PA$  and  $PB_j$  ( $j = 1, \dots, r$ ) to  $\mathcal{C}^\infty$  vector fields on  $\mathbb{R}^d \times \mathbb{R}^D$  with supports of the form  $\mathbb{R}^d \times \mathcal{K}$ ,  $\mathcal{K}$  compact. Let us denote by  $(Z_t)$  the solution of the corresponding stochastic differential system. It is well known (cf. Kunita [10] e.g.) that the stochastic flow  $\eta_t(z)$



associated to this system satisfies : for any integer  $i$  ( $i = 1, \dots, d$ ), there exists a random variable  $L_i^t$ , having moments of all orders, such that :

$$|\partial_i \eta_t(z)| \leq L_i^t(1 + |z|^2)$$

We also extend the function  $f$  to a  $\mathcal{C}^\infty$  function  $g$  on  $\mathbb{R}^d \times \mathbb{R}^D$ , with a support of the form  $\mathbb{R}^d \times \mathcal{K}_1$ ,  $\mathcal{K}_1$  compact. Let us define  $\zeta(t, z) = Eg(Z_t(z))$ . We then have, for any multi-index  $I$  and some constants  $n_I, C_I(t)$  :

$$|\partial_I \zeta(t, z)| \leq C_I(t)(1 + |z|^{n_I})$$

The restriction of  $\zeta(t, z)$  to the cartesian product of  $\mathbb{R}^d$  and the imbedding of  $\mathbb{P}^{d-1}$  is  $u(t, \theta)$  ; expressing  $\theta$  in the local coordinates given by a large charts atlas on  $\mathbb{P}^{d-1}$ , we then deduce (15).

□

### Remarks

- In the sequel, we will not go on working with the process  $(Z_t)$ , because the associated differential operator is not strongly elliptic. Thus we cannot avoid working on  $\mathbb{P}\mathbb{R}^d$ .
- For any integer  $s$ , let us now define :

$$\pi_s(x) = \frac{1}{(1 + |x|^2)^s}$$

The previous Lemma shows that, for any integer  $n \geq 0$ , there exists an integer  $s_n$  such that, for any multi-index  $I$  of length  $l(I)$  smaller than  $n$ , for any  $t \geq 0$  :

$$|\partial_I u(t, \phi_\theta^{-1}(x, y))| \pi_{s_n}(x) \in L^2(\mathbb{R}^d \times Val(\phi_\theta)) \quad (16)$$

This will be often implicitly used in the sequel to justify the existence of integrals w.r.t.  $d\nu$  (we recall that  $\nu$  denotes the unique invariant probability law of  $(x_t, [v_t])$ ) or measures of type  $\pi_s(x)dx$ .

**Without loss of generality, thereafter we will assume :**

$$\int_{\mathbb{P}\mathbb{R}^d} f(s) d\nu(s) = 0 \quad (17)$$

(if it is not the case, we change  $f$  in  $\tilde{f} = f - \int_{\mathbb{P}\mathbb{R}^d} f(s) d\nu(s)$ ).

Now, we will adapt for  $u(t, \theta)$  the method used in [13] to establish some exponential decay results on the function  $(t, x) \rightarrow Eg(x_t(x))$  and its derivatives.

We begin by an easy Proposition, proved in [13].

**Proposition 7.2** (i) For any integer  $n$  :

$$\exists C_n > 0, \exists \gamma_n > 0 : E|x_t(x)|^n \leq C_n(1 + |x|^n \exp(-\gamma_n t)) \quad , \quad \forall t, \forall x \quad (18)$$

(ii) The unique invariant probability measure of  $(x_t)$ ,  $\mu$ , has a smooth density  $p(x)$  and finite moments of any order.

This implies that any function  $f$  satisfying the requirements of the Lemma (6.3) belongs to  $L^2(P\mathcal{M}; d\nu)$ .

The plan of the sequel of the proof is the following (in all the inequalities which follow, the constants must be understood strictly positive) :

• we show :

$$\forall t > 0 \quad , \quad \int_{\mathbb{P}\mathbb{R}^d} |u(t, \theta)|^2 d\nu(\theta) \leq C \exp(-\kappa t)$$

• this inequality permits to show that, for some family of differential operators  $L_k$  defined below, we have :

$$\int_{\mathbb{P}\mathbb{R}^d} |L_{k_1} \dots L_{k_q} u(t, \theta)|^2 d\nu(\theta) \leq C_q \exp(-\kappa_q t)$$

• then we use these estimates to prove that, for  $s$  large enough :

$$\forall t > 0 \quad , \quad \int |u(t, x)|^2 \pi_s(x) dx \leq C \exp(-\lambda t)$$

•  $d\rho([v])$  denoting the trace of  $d\tau(x, [v])$  on  $\mathbb{P}^{d-1}$ , the previous inequality will permit to get :

$$\int_{\mathbb{P}\mathbb{R}^d} |L_{k_1} \dots L_{k_q} u(t, \theta)|^2 \pi_s(x) dx \otimes d\rho([v]) \leq C_q \exp(-\lambda_q t)$$

• finally, we get :

$$\int_{\mathbb{P}\mathbb{R}^d} |L_{k_1} \dots L_{k_q} u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq C_J \exp(-\gamma_J t)$$

Expressing this inequality in local coordinates, we use the Sobolev imbedding theorem to conclude.

## 7.2 Upper bounds in $L^2(\mathbb{P}\mathbb{R}^d; d\nu(\theta))$

We now state a result, which is an immediate extension of the Lemma 6.1 of [13] (one just needs to remark that the hypothesis  $(H2)(i)$  implies that  $\nu$ , the unique invariant probability law of  $(x_t, [v_t])$ , has a strictly positive smooth density).

**Lemma 7.3** *Under the hypotheses of Theorem (6.1) and (17), there exist strictly positive constants  $C$  and  $\kappa$  such that :*

$$\forall t > 0, \quad \int_{\mathbb{P}\mathbb{R}^d} |u(t, \theta)|^2 d\nu(\theta) \leq C \exp(-\kappa t) \quad (19)$$

Let us define some differential operators on  $\mathbb{P}\mathbb{R}^d$  by :

$$L_j = \sum_{i=1}^d (PB_j)^i(\cdot) \partial_i$$

Now, remembering that  $L$  is the differential operator associated to  $(x_t, v_t)$ , we are equipped to prove :

**Lemma 7.4** *Under (H1), (H2) and (17), there exist strictly positive constants  $C_1$  and  $\kappa_1$  such that :*

$$\sum_{k=1}^r \int_{\mathbb{P}\mathbb{R}^d} L_k |u(t, \theta)|^2 d\nu(\theta) \leq C_1 \exp(-\kappa_1 t) \quad (20)$$

Proof of 7.4 remark :

$$\frac{d}{dt} |u(t, \theta)|^2 - L |u(t, \theta)|^2 = - \sum_{k=1}^r (L_k u(t, \theta))^2$$

Let us choose  $0 < \delta < \kappa$ . Multiplying the previous equality by  $e^{\delta t}$ , integrating with respect to  $d\nu$ , using  $L^* \nu = 0$  :

$$e^{\delta t} \frac{d}{dt} \int_{\mathbb{P}\mathbb{R}^d} |u(t, \theta)|^2 d\nu(\theta) + e^{\delta t} \sum_{k=1}^r \int_{\mathbb{P}\mathbb{R}^d} |L_k u(t, \theta)|^2 d\nu(\theta) \leq 0$$

Now, let us choose an arbitrarily large time  $T$  and integrate from 0 to  $T$  the previous inequality ; we obtain :

$$\begin{aligned} & e^{\delta T} \int_{\mathbb{P}\mathbb{R}^d} |u(T, \theta)|^2 d\nu(\theta) + \int_0^T e^{\delta t} \left( \sum_{k=1}^r \int_{\mathbb{P}\mathbb{R}^d} |L_k u(t, \theta)|^2 d\nu(\theta) \right) dt \\ & \leq \int_{\mathbb{P}\mathbb{R}^d} |f(\theta)|^2 d\nu(\theta) + \delta \int_0^T e^{\delta t} \left( \int_{\mathbb{P}\mathbb{R}^d} |u(t, \theta)|^2 d\nu(\theta) \right) dt \end{aligned}$$

Thus, using (19) :

$$\sum_{k=1}^r \int_0^{+\infty} e^{\delta t} \left( \int_{\mathbb{P}\mathbb{R}^d} |L_k u(t, \theta)|^2 d\nu(\theta) \right) dt < +\infty$$

Then, we remark that there exist strictly positive constants  $C_2$ ,  $C_3$  and  $C_4$  such that :

$$\begin{aligned} \frac{d}{dt} \sum_{k=1}^r |L_k u(t, \theta)|^2 - L \sum_{k=1}^r |L_k u(t, \theta)|^2 \\ \leq -C_2 \sum_{k,l=1}^r |L_k L_l u(t, \theta)|^2 + (C_3|x| + C_4) \sum_{k=1}^r |L_k u(t, \theta)|^2 \end{aligned}$$

Let us choose  $0 < \kappa_1 < \delta$ . Multiplying the previous equality by  $e^{\kappa_1 t}$ , integrating with respect to  $d\nu(\theta)$ , and then with respect to  $\xi$  from 0 to  $t$ , we obtain, for some strictly positive constant  $C_1$  :

$$\sum_{k=1}^r \int_{\mathbb{P}\mathbb{R}^d} |L_k u(t, \theta)|^2 d\nu(\theta) \leq C_1 e^{-\kappa_1 t}$$

That ends the proof.

□

**Corollary 7.5** *Under (H1), (H2) and (17), for any integer  $q$ , there exist strictly positive constants  $C_q$  and  $\kappa_q$  such that, for any  $k_1, \dots, k_q$  in  $\{1, \dots, r\}$ , we have :*

$$\int_{\mathbb{P}\mathbb{R}^d} |L_{k_1} \dots L_{k_q} u(t, \theta)|^2 d\nu(\theta) \leq C_q \exp(-\kappa_q t) \quad (21)$$

**Proof** One can proceed by a recurrence over  $q$ , and performing the same kind of integrations as before, starting from the inequality (where  $C_J$  and  $C_q$  are some strictly positive constants) :

$$\begin{aligned} \frac{d}{dt} |L_{k_1} \dots L_{k_q} u(t, \theta)|^2 - L |L_{k_1} \dots L_{k_q} u(t, \theta)|^2 \\ \leq -C_q \sum_{j=1}^r L_j |L_{k_1} \dots L_{k_q} u(t, \theta)|^2 + \sum_{m=1}^q \sum_{J=(j_1 \dots j_m)} C_J (|x| + 1) |L_{j_1} \dots L_{j_m} u(t, \theta)|^2 \end{aligned}$$

□

We recall that for any integer  $s$ , we have defined :

$$\pi_s(x) = \frac{1}{(1 + |x|^2)^s}$$

and that  $d\rho([v])$  denotes the trace of  $d\tau(x, [v])$  on  $\mathbb{P}^{d-1}$ .

### 7.3 Upper bounds in $L^2(\mathbb{P}\mathbb{R}^d; \pi_s(x)dx \otimes d\rho([v]))$

**Lemma 7.6** *Under the hypotheses (H1), (H2) and (17), there exist strictly positive constants  $C$  and  $\lambda$  such that*

$$\forall t > 0 \quad , \quad \int_{\mathbb{P}\mathbb{R}^d} |u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq C \exp(-\lambda t) \quad (22)$$

**Proof** As, for any multi-index  $J$ , we have :

$$\partial_J \pi_s(x) = \psi_J(x) \pi_s(x) \quad , \quad \psi_J(x) \text{ bounded function} \quad (23)$$

we remark that there exist an integer  $s$  and functions  $\phi_1(x)$  and  $\phi_2(x)$  such that :

- $\phi_1(x)$  is a bounded function independent of  $s$  ;
- $\phi_2(x)$  is a function depending on  $s$ , but tending to 0 when  $|x| \rightarrow +\infty$  ;
- the following inequality holds :

$$\begin{aligned} & \int_{\mathbb{P}\mathbb{R}^d} |u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \\ & \leq \int_{\mathbb{P}\mathbb{R}^d} \left( \phi_1(x, [v]) + \phi_2(x, [v]) + s \frac{\langle A(x), x \rangle}{1 + |x|^2} \right) |u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \\ & \quad - C \sum_{k=1}^r \int |L_k u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \end{aligned}$$

After having possibly increased the value of  $s$ , we can choose a ball  $B = B(0, R_0)$  such that :

$$\forall x \in \mathbb{R}^d - B \quad , \quad \forall v \quad , \quad \phi_1(x, [v]) + \phi_2(x, [v]) + s \frac{\langle A(x), x \rangle}{1 + |x|^2} < -1 \quad (24)$$

We divide the previous integral over  $\mathbb{P}\mathbb{R}^d$ , on integrals over  $B$  and the complementary set of  $B$ . The estimation (19) permits to get an upper bound of the form  $C \exp(-\lambda t)$  for the integration on  $B$ , since  $\nu$  has a smooth and strictly positive density.

An easy computation then shows that, for some strictly positive constant  $C$  :

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{P}\mathbb{R}^d} |u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq - \int |u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) + C \exp(-\lambda t)$$

That ends the proof.

□

Now, we remark that there exist an integer  $s$  and functions  $\varphi_1(x)$  and  $\varphi_2(x)$  such that :

- $\varphi_1(x)$  is a bounded function independent of  $s$  ;

- $\varphi_2(x)$  is a function depending on  $s$ , but tending to 0 when  $|x| \rightarrow +\infty$  ;
- the following equality holds :

$$\begin{aligned}
& \int_{\mathbb{P}\mathbb{R}^d} L|u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \\
&= \int_{\mathbb{P}\mathbb{R}^d} |u(t, x, [v])|^2 L^* \pi_s(x) dx \otimes d\rho([v]) \\
&= \int_{\mathbb{P}\mathbb{R}^d} \left( \varphi_1(x) + \varphi_2(x) + 2s \frac{\langle A(x), x \rangle}{1 + |x|^2} \right) |u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v])
\end{aligned}$$

Therefore, similar arguments as those used for the previous Lemma and (22) show that there exist an integer  $s$  and strictly positive constant  $C_1, \lambda_1$  satisfying :

$$\int_{\mathbb{P}\mathbb{R}^d} L|u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq C_1 \exp(-\lambda_1 t) \quad (25)$$

As well, using also arguments employed in the previous Section, we can prove the existence of strictly positive constants  $\lambda_q$  and  $C_q$  such that, for any  $k_1, \dots, k_q$  in  $\{1, \dots, r\}$ , we have :

$$\int_{\mathbb{P}\mathbb{R}^d} L \left( |L_{k_1} \dots L_{k_q} u(t, \theta)|^2 \right) \pi_s(x) dx \otimes d\rho([v]) \leq C_q \exp(-\lambda_q t) \quad (26)$$

## 7.4 End of the Proof of Lemma (6.4)

We remark again :

$$\frac{d}{dt} |u(t, \theta)|^2 - L|u(t, \theta)|^2 = - \sum_{k=1}^r (L_k u(t, \theta))^2$$

Let us choose  $0 < \delta < \lambda_0$ . Multiplying the previous equality by  $e^{\delta t}$ , integrating with respect to  $\pi_s(x) dx \otimes d\rho([v])$ , we get, using (25) :

$$\begin{aligned}
& e^{\delta t} \frac{d}{dt} \int_{\mathbb{P}\mathbb{R}^d} |u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) + e^{\delta t} \sum_{k=1}^r \int_{\mathbb{P}\mathbb{R}^d} (L_k u(t, x, [v]))^2 \pi_s(x) dx \otimes d\rho([v]) \\
& \leq C_0 e^{-(\lambda - \delta)t}
\end{aligned}$$

Now, let us choose an arbitrarily large time  $T$  and integrate from 0 to  $T$  the previous inequality ; we obtain, for some strictly positive constants  $C_1$  and  $s$  large enough :

$$\begin{aligned}
& e^{\delta T} \int_{\mathbb{P}\mathbb{R}^d} |u(T, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) + \int_0^T e^{\delta t} \left( \sum_{k=1}^r \int_{\mathbb{P}\mathbb{R}^d} |(L_k u(t, x, [v]))|^2 \pi_s(x) dx \otimes d\rho([v]) \right) dt \\
& \leq \int_{\mathbb{P}\mathbb{R}^d} |f(x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) + \delta \int_0^T e^{\delta t} \left( \int_{\mathbb{P}\mathbb{R}^d} |u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \right) dt + C
\end{aligned}$$

Thus :

$$\sum_{k=1}^r \int_0^{+\infty} e^{\delta t} \left( \int_{\mathbb{P}\mathbb{R}^d} |L_k u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \right) dt < +\infty$$

If necessary, we increase the value of  $s$ , in order to obtain that, for any constants  $D_1$  and  $D_2$  :

$$\sum_{k=1}^r \int_0^{+\infty} e^{\delta t} \left( \int_{\mathbb{P}\mathbb{R}^d} (D_1|x| + D_2) |L_k u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \right) dt < +\infty \quad (27)$$

Then, we remark that there exist strictly positive constants  $C_2$ ,  $C_3$  and  $C_4$  such that :

$$\begin{aligned} & \frac{d}{dt} \left( \sum_{k=1}^r |L_k u(t, x, [v])|^2 \right) - L \left( \sum_{k=1}^r |L_k u(t, x, [v])|^2 \right) \\ & \leq -C_2 \sum_{k,l=1}^r |L_k L_l u(t, x, [v])|^2 + (C_3|x| + C_4) \sum_{k=1}^r |L_k u(t, x, [v])|^2 \end{aligned}$$

Let us choose  $0 < \delta_1 < \delta$ . Multiplying the previous equality by  $e^{\delta_1 t}$ , integrating with respect to  $\pi_s(x) dx \otimes d\rho([v])$ , and then with respect to  $\xi$  from 0 to  $t$ , using (26) and (27), we obtain for some strictly positive constant  $C_5$  :

$$\sum_{k=1}^r \int_{\mathbb{P}\mathbb{R}^d} |L_k u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq C_5 e^{-\delta_1 t}$$

From the previous inequality and (H2)(i), we deduce the existence of strictly positive constants  $C$ ,  $\gamma$ , independent on  $\theta = (x, [v])$ , such that :

$$\int_{Val(\phi_\theta)} |\nabla u(t, \phi_\theta^{-1}(x, y))|^2 \pi_s(x) dx dy \leq C e^{-\gamma t}$$

By the same procedure, one may show that, for any sequence  $J = (k_1, \dots, k_q)$  of integers in  $1, \dots, r$ , there exist strictly positive constants  $C_J$  and  $\gamma_J$  such that :

$$\int_{\mathbb{P}\mathbb{R}^d} |L_{k_1} \dots L_{k_q} u(t, x, [v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq C_J \exp(-\gamma_J t)$$

We deduce that for any multi-index  $J$  (referring to derivatives w.r.t. the coordinates  $(x, y)$ ), there exist strictly positive constants  $C_J$  and  $\gamma_J$ , independent on  $\theta$ , such that :

$$\int_{Val(\phi_\theta)} |\partial_J u(t, \phi_\theta^{-1}(x, y))|^2 \pi_s(x) dx dy \leq C_J \exp(-\gamma_J t)$$

We already have remarked that for any multi-index  $J$  :

$$\partial_J \pi_s(x) = \psi_J(x) \pi_s(x) \quad , \quad \psi_J(x) \text{ bounded}$$

Therefore we get that, for any multi-index  $I$ , any integer  $M$ , there exist an integer  $s$  and strictly positive constants  $C$  and  $\lambda$  such that :

$$\forall m \leq M, \forall t > 0, \int_{Val(\phi_\theta)} |D^m(u(t, x, y)\pi_s(x))|^2 dx dy \leq C \exp(-\lambda t)$$

so that we deduce (15) as a consequence of the Sobolev imbedding Theorem (in (3.1), we have constrained  $Val(\phi_\theta)$  to be a convex domain).

## 8 Approximation of the spectrum

The approximation of all the Lyapunov exponents may be performed in the following way.

For  $x$  in  $\mathcal{M}$ , and any integer  $k$  ( $1 \leq k \leq d$ ), let  $G_k(T_x \mathcal{M})$  be the Grassmann manifold of all  $k$ -dimensional subspaces of the tangent space  $T_x \mathcal{M}$ , and  $G_k(T\mathcal{M})$  be the Grassmann bundle  $\cup_{x \in \mathcal{M}} G_k(T_x \mathcal{M})$ .

For any  $K$  in  $G_k(T_x \mathcal{M})$ , let  $K_t = T x_t(x) K \subset T_{x_t(x)} \mathcal{M}$ , and  $J_t(K) = |\det(T x_t(x)|_K)|$ .

Let us suppose that the above hypothesis (H1) holds and that the infinitesimal generators of the processes  $(K_t)$  are strongly elliptic.

In this context (and under even weaker hypotheses : Baxendale [4]), for any  $1 \leq k \leq d$ , there exist real numbers  $\lambda_1, \dots, \lambda_k$  and a smooth function  $\psi_k$  on  $G_k(T\mathcal{M})$  such that, for any  $x$  in  $\mathcal{M}$  and  $K$  in  $G_k(T_x \mathcal{M})$ , if  $\nu$  now denotes the unique invariant probability law of  $(K_t)$  :

$$a.s. \lim_{t \rightarrow +\infty} \frac{1}{t} \log J_t(K) = \lambda_1 + \dots + \lambda_k = \int_{G_k(T\mathcal{M})} \psi_k(\theta) \nu(d\theta)$$

To approximate the exponents, we first choose  $k = 1$  and approximate  $\lambda_1 = \lambda$  as described in the previous Sections ; then we choose an orthonormal basis  $(v_0^1, v_0^2)$  of a sub-space  $K$  of  $T_{x_0} \mathcal{M}$  for initial value of  $\bar{v}_p^h$ , and we compute

$$\frac{1}{N} \log |\det(\bar{v}_N^{h,1}, \bar{v}_N^{h,2})| \quad (28)$$

Going on up to  $d$ , we successively get approximations of all the sums  $\lambda_1 + \dots + \lambda_k$ . This provides the approximate values of the exponents. With the same technique as above, we may prove that the approximation error on each of these sums is of order  $h$ , and therefore the error on each  $\lambda_k$  is also of order  $h$ .

## 9 Numerical tests

The Fortran programs corresponding to the two examples below, have been generated by Presto, a software of automatic generation of programs for the simulation of S.D.E.'s [14].



## 9.1 System in $\mathbb{R}^d$

We choose  $d = r = 1$ , and consider the following system, in the Ito sense :

$$dx_t = (-ax_t + F(x_t))dt + G(x_t)dW(t)$$

Linearizing it, it is easy to see that, for  $a < 0$  and  $F, G$  with continuous bounded derivatives, if  $G(x) > G_0 > 0$  for all  $x$ , then the upper Lyapunov exponent of the system exists and is given by :

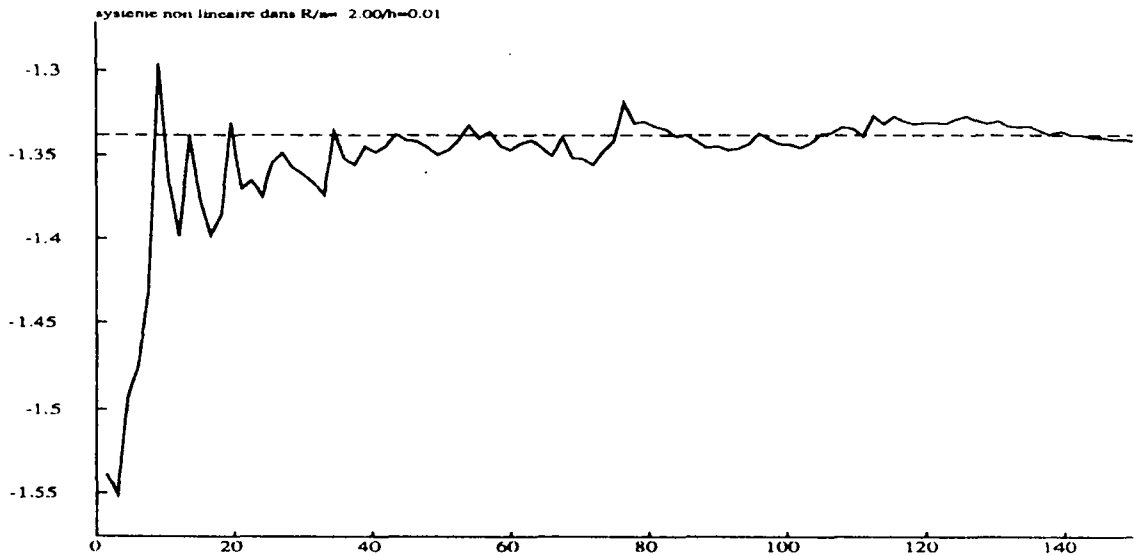
$$\lambda = -a + \int_{\mathbb{R}} (F'(x) - \frac{1}{2}G'^2(x))p(x)dx$$

where  $p(x)$  is the density of the unique invariant probability law of  $(x_t)$ .

We then choose :  $F(x) = \arctan(x)$  and  $G(x) = \sqrt{1+x^2}$ .

Solving the stationary Fokker-Planck equation, we get the explicit expression of the unnormalized stationary density. For each value of  $a$ , we can compute numerically the normalization constant (just integrating over  $\mathbb{R}$  the unnormalized density). This permits us to compute the “true” value of  $\lambda$ .

For example, for  $a = 2$ , the normalization constant approximately is 1.491, and  $\lambda = -1.3385$ . The figure below shows the time evolution of  $\bar{\lambda}_p^h$  (in the x-axis :  $ph$ ), for  $h = 0.01$ .



## 9.2 System in $\mathcal{M}$

Let us consider the stochastic differential system in the Stratonovich sense on  $\mathcal{M} = \mathcal{S}^1$  :

$$\begin{aligned} d\varphi_t &= \sin\varphi_t \circ dW_1(t) + \cos\varphi_t \circ dW_2(t) \\ \varphi_0 &= 0 \end{aligned} \quad (29)$$

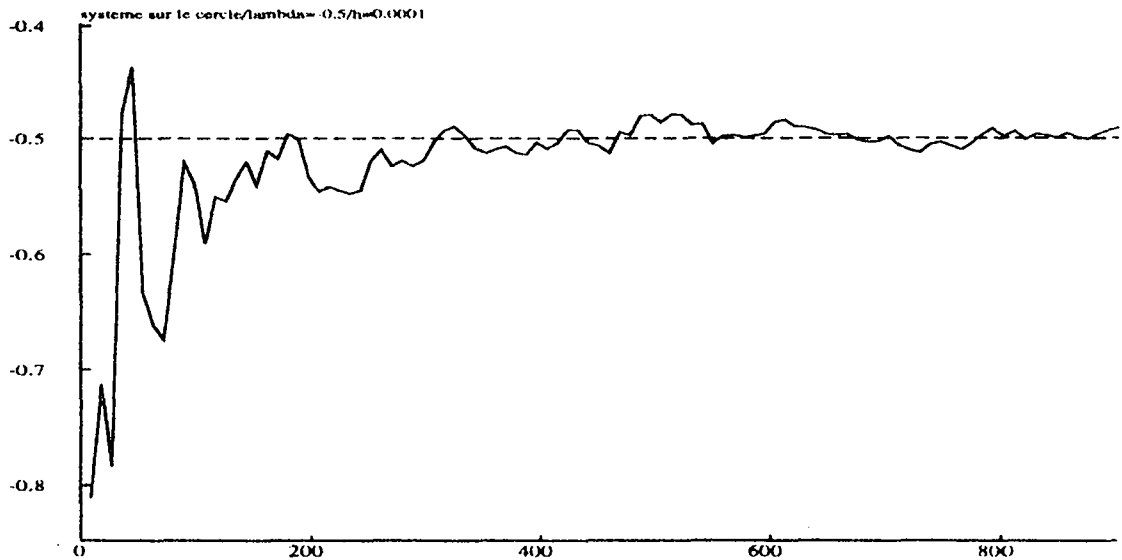
We know (Cf. Arnold [1]) that the infinitesimal operator is  $L = -\frac{1}{2}\Delta$ , thus the motion is a Brownian motion on the sphere and the upper Lyapunov exponent is  $\lambda = -\frac{1}{2}$ .

Considering  $\mathcal{S}^1$  imbedded in  $\mathbb{R}^2$ , let us denote  $x_t = (x_1(t), x_2(t)) = (\cos\varphi_t, \sin\varphi_t)$  the image of the process in  $\mathbb{R}^2$ . We choose the atlas  $\mathcal{A}$  defined by two charts  $\phi_1$  and  $\phi_2$ , which are the stereographic projections respectively w.r.t the north pole  $(0, 1)$ , and the south pole  $(0, -1)$ . We define  $Dom(\phi_1)$  (resp.  $Dom(\phi_2)$ ) as the points whose angular angle belongs to  $[\frac{3\pi}{4}; \frac{5\pi}{4}]$  (resp.  $[-\frac{3\pi}{4}; -\frac{\pi}{4}]$ ) :  $\mathcal{A}$  is a large charts atlas.

Let  $X_t$  denote the process on  $\mathbb{R}$  image of  $x_t$  by  $\phi_1$  ; the linearized process  $(X_t, V_t)$  satisfies the following stochastic differential equation, with  $\varepsilon = 1$  (in the local coordinates corresponding to  $\phi_2$ , the system differs only by  $\varepsilon = -1$ ) :

$$\begin{aligned} dX_t &= \frac{1}{4} \left( 1 + \frac{X_t^2}{4} \right) \left( \sin \left( 4 \arctan \frac{X_t}{2} \right) - X_t \cos \left( 4 \arctan \frac{X_t}{2} \right) \right) dt \\ &\quad + \left( 1 + \frac{X_t^2}{4} \right) \left( \varepsilon \sin \left( 2 \arctan \frac{X_t}{2} \right) dW_1(t) - \cos \left( 2 \arctan \frac{X_t}{2} \right) dW_2(t) \right) \\ dV_t &= \frac{1}{4} \left( \frac{5X_t}{2} \sin \left( 4 \arctan \frac{X_t}{2} \right) + \left( 1 - \frac{3X_t^2}{2} \right) \cos \left( 4 \arctan \frac{X_t}{2} \right) \right) V_t dt \\ &\quad + \varepsilon \left( \frac{X_t}{2} \sin \left( 2 \arctan \frac{X_t}{2} \right) + \cos \left( 2 \arctan \frac{X_t}{2} \right) \right) V_t dW_1(t) \\ &\quad + \left( \sin \left( 2 \arctan \frac{X_t}{2} \right) - \frac{X_t}{2} \cos \left( 2 \arctan \frac{X_t}{2} \right) \right) V_t dW_2(t) \\ (X_0, V_0) &= (0, 1) \end{aligned} \quad (30)$$

Below, we show the time evolution of  $\bar{\lambda}_p^h$  (in the x-axis :  $ph$ ), for  $h = 0.0001$ .



We observe that the simulation time to get a good approximation of  $\lambda$  is much larger than in the previous example. This is due to the fact that the system is less stable, since here the Lyapunov exponent is near 0.

## 10 Conclusion

We have proposed an algorithm of approximation of the Lyapunov exponents of non linear stochastic differential systems, and given a theoretical estimate of its convergence rate.

From a numerical point of view, as in the linear case [12], the pertinent choice of the number  $N$  of steps in the formula (28) may present important difficulties. Further studies in that direction are necessary.

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